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MacNeille Completions of *D*-Posets and Effect Algebras[†]

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Our main result is stating and proving a necessary and sufficient condition for *D*-poset and effect algebras to have MacNeille completions.

1. INTRODUCTION AND BASIC DEFINITIONS

Recently, new algebraic structures in the axiomatic approach to stochastic quantum mechanics have been introduced. They are a weakening of the axiomatic system of orthomodular lattices (or posets).

Kôpka [8] introduced a new algebraic structure of fuzzy sets—a *D*-poset of fuzzy sets. A difference of comparable fuzzy sets is a primary operation in this structure. Later, Kôpka and Chovanec [9], by transferring the properties of a difference operation of *D*-poset of fuzzy sets to an arbitrary partially ordered set, obtained a new algebraic structure—a *D*-poset—that generalized orthoalgebras and MV algebras.

In the sequel, for a partial operation $\oplus$ (or $\ominus$) on a set X and for a, b, $c \in X$, when we write $a \oplus b = c$ ($c \ominus b = a$), we mean that $a \oplus b$ ($c \ominus b$) is defined and $a \oplus b = c$ ($c \ominus b = a$).

Definition 1.1 [9] Let $(P, \leq)$ be a poset with the least element 0 and the greatest element 1. Let $\ominus$ be a partial binary operation on P such that $b \ominus a$ is defined iff $a \leq b$. Then $(P; \leq, \ominus, 0, 1)$ is called a *difference poset* (*D*-poset) if the following conditions are satisfied:

[†] This paper is dedicated to the memory of Prof. Gottfried T. Rüttimann.

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(Di) For any $a \in P$, $a \ominus 0 = a$.
(Dii) If $a \leq b \leq c$, then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

Effect algebras (introduced by Foulis and Bennett [3]) are important for modeling unsharp measurements in Hilbert space: The set of all effects is the set of all self-adjoint operators T on a Hilbert space H with $0 \leq T \leq 1$. In a general algebraic form an effect algebra is defined as follows:

Definition 1.2. A structure $(E; \oplus, 0, 1)$ is called an *effect-algebra* if 0, 1 are two distinguished elements and $\oplus$ is a partially defined binary operation on P which satisfies the following conditions for any $a, b, c \in E$:

(Ei) $b \oplus a = a \oplus b$ if $a \oplus b$ is defined.
(Eii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ if the expression on either side is defined.
(Eiii) For every $a \in P$ there exists a unique $b \in P$ such that $a \oplus b = 1$.
(Eiv) If $1 \oplus a$ is defined, then $a = 0$.

Note that we denote the unique element $b \in P$ in (Eiii) by a', hence $a \oplus a' = 1$ for all $a \in P$.

We can easily prove the following statement.

Proposition 1.3 (*Cancellation Properties*):

(a) In a D-poset $(P; \leq, \ominus, 0, 1)$, for all $a, b, c \in P$ with $a \leq b$, $a \leq c$ we have

$$b \ominus a = c \ominus a \qquad \text{implies} \quad b = c$$

(b) In an effect algebra $(E; \oplus, 0, 1)$, for all $a, b, c \in E$ with defined $a \oplus b$ and $a \oplus c$ we have

$$a \oplus b = a \oplus c \qquad \text{implies} \quad b = c$$

Corollary 1.4:

(1) In every D-poset $(P; \leq, \ominus, 0, 1)$ the partial binary operation $\oplus$ can be defined by setting

(EP) $a \oplus b$ is defined and $a \oplus b = c$ iff $a \leq c$ and $c \ominus a = b$

(2) In every effect algebra E the partial binary operation $\ominus$ and the relation $\leq$ can be defined by setting

(PE) $a \leq c$ and $c \ominus a = b$ iff $a \oplus b$ is defined and $a \oplus b = c$

Cancellation properties guarantee that $\oplus$, $\ominus$, and $\leq$ are well defined. Moreover, it is easy to show that the following statements hold:

Proposition 1.5. (1) In every D-poset $(P; \leq, \ominus, 0, 1)$, the partial binary operation $\oplus$ derived by (EP) fulfils the axioms (Ei)–(Eiv) of effect algebra.

(2) In every effect algebra $(E, \oplus, 0, 1)$, the partial binary operation $\ominus$ and the partial order $\leq$ deemed by (PE) fulfil the axioms (Di)–(Dii) of D-poset.

A D-algebra is a generalization of a D-poset in which a partial order is not assumed. However, if a D-algebra is equipped with a natural partial order (derived from the partial operation $\ominus$), then it becomes a D-poset. We present here the definition of D-algebra by Gudder [5].

Definition 1.6. A partial algebra $(P; \ominus, 0, 1)$ is called a *D-algebra* if 0, 1 are two distinguished elements of P; and $\ominus$ is a partially defined binary operation on P which satisfies the following conditions ($a, b, c \in P$):

- (Ai) $a \ominus 0$ is defined and $a \ominus 0 = a$ for all $a \in P$.
- (Aii) $1 \ominus a$ is defined for all $a \in P$.
- (Aiii) If $0 \ominus a$ is defined, then $a = 0$.
- (Aiv) If $b \ominus a$ and $c \ominus b$ are defined, then $c \ominus a$ and $(c \ominus a) \ominus (c \ominus b)$ are defined and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

We can easily show the following statements:

Proposition 1.7:

- (i) If $(P; \leq, \ominus, 0, 1)$ is a D-poset, then $(P; \ominus, 0, 1)$ is a D-algebra.
- (ii) If $(P; \ominus, 0, 1)$ is a D-algebra and we define $a \leq b$ iff $b \ominus a$ is defined, then the relation $\leq$ is a partial order on P and $(P; \leq, \ominus, 0, 1)$ is a D-poset.

See ref. 5 for the proof.

Corollary 1.8. In every D-algebra $(P; \oplus, 0, 1)$ for $a, b, c \in P$ the cancellation property $b \ominus a = c \ominus a$ implies $b = c$ is satisfied and the partial binary operation $\oplus$ can be defined by:

(EA) $a \oplus b$ is defined and $a \oplus b = c$ iff $c \ominus a$ is defined and $c \ominus a = b$.

Then $(P; \oplus, 0, 1)$ is an effect algebra.

For more details we refer the reader to refs. 3, 5 and 9.

In view of the above observations, every effect algebra (D-algebra) is ordered in a natural way. An effect algebra (D-algebra, D-poset) is called a *lattice effect algebra* (D-algebra, D-lattice) if, as a poset, it forms a lattice, and it is called *complete* if supremum and infimum of every subset exist.

It is well known that every poset has a MacNeille completion (i.e., completion by cuts [1]). By Schmidt [16], a *MacNeille completion* $MC(P)$

of a poset P is (up to isomorphism) any complete lattice into which P can be supremum-densely and infimum-densely embedded [i.e., for every element $x \in MC(P)$ there exist $M, Q \subseteq P$ such that $x = \vee\varphi(M) = \wedge\varphi(Q)$, where φ: $P \to MC(P)$ is the embedding]. We usually identify P with $\varphi(P)$. In this sense $MC(P)$ preserves all infima and suprema existing in P.

It is well known that a MacNeille completion of any Boolean algebra is a complete Boolean algebra and, similarly, a MacNeille completion of an ortholattice is a complete ortholattice, both understood with orthocomplementation extended to $MC(P)$ in a natural way [7]. On the other hand, a MacNeille completion of a modular ortholattice or orthomodular lattice need not be orthomodular [7]. Some conditions for positive results are known.

In ref. 14 examples of effect algebras (D-posets) which cannot be densely embedded into any complete effect algebras (D-algebras) were introduced. The aim of this paper is to characterize those effect algebras (D-algebras) which can be supremum-densely and infimum-densely embedded into complete effect algebras (D-algebras), calling the latter MacNeille completions.

2. STRONGLY *D*-CONTINUOUS *D*-ALGEBRAS

In the sequel, for subsets U, Q of a D-algebra $(P; \ominus, 0, 1)$, we will write $U \le Q$ iff $u \le q$ for all $u \in U$, $q \in Q$. In such a case we will write $Q \ominus U = \{q \ominus u | q \in Q, u \in U\}$.

Definition 2.1. A D-algebra $(P; \ominus, 0, 1)$ is called *strongly D-continuous* iff for all $U, Q \subseteq P$ with $U \le Q$ the following condition is satisfied:

(SDC) $\wedge(Q \ominus U) = 0$ iff every lower bound of Q is under every upper bound of U.

Definition 2.2. A D-algebra $(P; \ominus, 0, 1)$ is called *complete* iff every subset of P has a supremum and an infimum.

Theorem 2.3. Every complete D-algebra is strongly D-continuous.

Proof. Suppose that $(E; \ominus, 0, 1)$ is a complete D-algebra. Let $U, Q \subseteq E$ with $U \le Q$.

(1) Let $\vee U = \wedge Q$. For any $x \le \wedge(Q \ominus U) = \wedge\{q \ominus u | q \in Q, u \in U\}$ and any $q \in Q$ we have $x \le q \ominus u$ for all $u \in U$. It follows that $u \le q \ominus x$ and hence $\vee U \le q \ominus x$, which implies that $x \oplus \vee U \le q$. We obtain that $x \oplus \vee U \le \wedge Q$ or equivalently $x \le (\wedge Q) \ominus (\vee U) = 0$. We conclude that $\wedge(Q \ominus U) = 0$.

(2) Assume that $\wedge(Q \ominus U) = 0$. Let us denote $a = \vee U$, $c = \wedge Q$. The assumption $U \le Q$ implies that $\vee U \le \wedge Q$, hence $a \le c$. Let $b = c \ominus a$. Then for every $q \in Q$ we have $a \oplus b = c \le q$ and thus $u \oplus b \le a \oplus b \le$

q for all $u \in U$. It follows that $b \leq q \ominus u$ for all $u \in U$, $q \in Q$, which implies that $b \leq \wedge(Q \ominus U) = 0$. We obtain that $c \ominus a = 0$, which implies $\wedge Q = \vee U$.

The following example shows that a noncomplete D-algebra need not be strongly D-continuous.

Example 2.4. Consider the D-algebra $(P; \ominus, 0, 1)$, where $P = \{0, a, b, c, d, 1\}$, and the partial binary operation $\ominus$ on P is defined in the following way:

$$c \ominus a = b, \qquad c \ominus b = a$$
$$d \ominus a = a, \qquad d \ominus b = b$$
$$1 \ominus c = a, \qquad 1 \ominus a = c$$
$$1 \ominus d = b, \qquad 1 \ominus b = d$$
$$x \ominus 0 = x \quad \text{for every} \quad x \in P$$

and for $x, y \in P$ we define $x \leq y$ iff $y \ominus x$ is defined. For subsets $U = \{a, b\}$ and $Q = \{c\}$ of P we have $U \leq Q$ and $Q \ominus U = \{a, b\}$. Clearly, $\wedge(Q \ominus U) = 0$. But c is a lower bound of Q which is not under the upper bound d of U. We conclude that P is not strongly D-continuous. Note that $\vee\{a, b\}$ does not exist, hence P is not a complete D-algebra.

Using the Schmidt characterization of the MacNeille completion of a poset, we can easily see the following assertion:

Lemma 2.5. Assume that $(P; \leq)$ is a poset. For all $U, Q \subseteq P$ with $U \leq Q$ the following conditions are equivalent:

(i) In a poset P every lower bound of Q is under every upper bound of U.
(ii) $\vee U = \wedge Q$ in the MacNeille completion $MC(P)$ of P.

Theorem 2.6. Let $(P; \ominus, 0, 1)$ be a strongly D-continuous D-algebra. Then for all $A, B, C \subseteq P$ with $A \leq B$ and $A \leq C$ the following condition in $MC(P)$ is satisfied:

(CL) $\quad \wedge(B \ominus A) = \wedge(C \ominus A)$ iff $\wedge B = \wedge C$.

Proof. Suppose that $A, B, C \subseteq P$ with $A \leq B$ and $A \leq C$.

(1) Assume first that $\wedge(B \ominus A) = \wedge(C \ominus A)$ in $MC(P)$. Then there exists $Z \subseteq P$ such that $\vee Z = \wedge(B \ominus A) = \wedge(C \ominus A)$ in $MP(C)$. By (SDC) we obtain that $\wedge((B \ominus A) \ominus Z) = \wedge((C \ominus A) \ominus Z) = 0$. Moreover, for all $a \in A$, $b \in B$, $c \in C$, and $z \in Z$ we have $(b \ominus a) \ominus z = b \ominus (a \oplus z)$ and $(c \ominus a) \ominus z = c \ominus (a \oplus z)$. Let $D = \{a \oplus z | a \in A, z \in Z\}$. Then $\wedge(B \ominus D) = \wedge(C \ominus D) = 0$, which by (SDC) implies that $\vee D = \wedge B = \wedge C$.

(2) Assume now that $\wedge B = \wedge C$. There exists $Z \subseteq P$ with $\vee Z = \wedge(B \ominus A)$ and by (SDC) we have $\wedge((B \ominus A) \ominus Z) = 0$. Let $D = \{a \oplus z | a \in A, z \in Z\}$. Then $\wedge(B \ominus D) = 0$, which by (SDC) implies that $\vee D = \wedge B = \wedge C$ and hence $\wedge((C \ominus A) \ominus Z) = \wedge(C \ominus D) = 0$. By (SDC) we obtain $\vee Z = \wedge(C \ominus A)$. We conclude that $\wedge(B \ominus A) = \wedge(C \ominus A)$.

3. COMPLETION OF STRONGLY *D*-CONTINUOUS *D*-ALGEBRAS

Throughout this section we assume that $(P; \ominus, 0, 1)$ is a strongly *D*-continuous *D*-algebra and $MC(P)$ is a MacNelle completion of a poset $(P; \leq)$, where $\leq$ on P is defined by $a \leq b$ iff $b \ominus a$ is defined. Moreover, we identify P with $\varphi(P)$, where $\varphi: P \to MC(P)$ is the embedding.

The aim of the section is to show that for a strongly *D*-continuous *D*-algebra $(P; \ominus, 0, 1)$ the operation $\ominus$ can be extended over $MC(P)$ in such a way that $MC(P)$ with constants 0, 1 and partial operation $\ominus$ [we will use the same symbol for operation on $MC(P)$] becomes a complete *D*-algebra.

Definition 3.1. For $a, c \in MC(P)$ with $a \leq c$ we put

$$c \ominus a = \wedge\{q \ominus u | q, u \in P, u \leq a, c \leq q\}$$

Note that for $a, c \in P$ with $a \leq c$ the definition of $c \ominus a$ in $MP(C)$ coincides with $c \ominus a$ in P. The proof that $(MC(P); \ominus, 0, 1)$ is a *D*-algebra is divided into a sequence of lemmas.

Lemma 3.2. For all $a, c \in MC(P)$ with $a \leq c$ it holds that $c \ominus a = 0$ iff $a = c$.

Proof. Let $U = \{u \in P | u \leq a\}$, $Q = \{q \in P | c \leq q\}$. Then $a = \vee U$, $c = \wedge Q$, and $U \leq Q$. By Definition 3.1, $c \ominus a = \wedge(Q \ominus U)$. In view of the strongly *D*-continuity of *P*, we obtain $c \ominus a = 0$ iff $a = c$.

Lemma 3.3. Let $a, b, c \in MC(P)$ be such that $a \leq c$ and $b \leq c \ominus a$. Then $a \leq c \ominus b$ and $(c \ominus b) \ominus a = (c \ominus a) \ominus b$.

Proof. Let $U_a = \{u \in P | u \leq a\}$, $U_b = \{v \in P | v \leq b\}$, and $Q = \{q \in P | c \leq q\}$. By the assumptions, for every $v \in U_b$ we have $v \leq b \leq c \ominus a = \wedge(Q \ominus U_a)$. It follows that for all $q \in Q$, $u \in U_a$, and $v \in U_b$ there exists $(q \ominus u) \ominus v$. By the properties of *D*-algebras the element $(q \ominus v) \ominus u$ exists and $(q \ominus u) \ominus v = (q \ominus v) \ominus u$. Thus $U_a \leq Q \ominus U_b$, which implies that $a = \vee U_a \leq \wedge(Q \ominus U_b) = c \ominus b$. Moreover,

$$\begin{aligned} &\wedge\{q \ominus u) \ominus v | u \in U_a, v \in U_b, q \in Q\} \\ &\quad = \wedge\{(q \ominus v) \ominus u) | u \in U_a, v \in U_b, q \in Q\} \end{aligned}$$

Let us show that $\wedge\{(q \ominus u) \ominus v | u \in U_a, v \in U_b, q \in Q\} = (c \ominus a) \ominus b$ and $\wedge\{(q \ominus v) \ominus u | u \in U_a, v \in U_b, q \in Q\} = (c \ominus b) \ominus a$. Let $D = \{s \in P | s \geq c \ominus a\}$. Then $\wedge D = c \ominus a = \wedge(Q \ominus U_a)$. By Theorem 2.6 we have $(c \ominus a) \ominus b = \wedge(D \ominus U_b) = \wedge((Q \ominus U_a) \ominus U_b)$. Similarly, if $E = \{e \in P | e \geq c \ominus b\}$, then $c \ominus b = \wedge E$ and $(c \ominus b) \ominus a = \wedge(E \ominus U_a) = \wedge((Q \ominus U_b) \ominus U_a)$. We conclude that $(c \ominus a) \ominus b = (c \ominus b) \ominus a$.

Lemma 3.4. For all $a, c \in MC(P)$ with $a \leq c$ it holds that $c \ominus (c \ominus a) = a$.

Proof. By Definition 3.1 there exists $c \ominus a$ and in view of Lemma 3.2, $(c \ominus a) \ominus (c \ominus a) = 0$. Using Lemma 3.3, we have $(c \ominus (c \ominus a)) \ominus a = 0$, which by Lemma 3.2 implies that $c \ominus (c \ominus a) = a$.

Lemma 3.5. For all $a, b, c \in MC(P)$ with $a \leq b \leq c$ there exists $(c \ominus a) \ominus (c \ominus b)$ and $b \ominus a = (c \ominus a) \ominus (c \ominus b)$.

Proof. By Definition 3.1 there exist $b \ominus a$ and $c \ominus b$. Using Lemma 3.4 and then Lemma 3.3 we obtain $b \ominus a = (c \ominus (c \ominus b)) \ominus a = (c \ominus a) \ominus (c \ominus b)$.

Important consequence of Lemmas 3.2–3.5 is the following assertion:

Theorem 3.6. For a strongly *D*-continuous *D*-algebra $(P; \ominus, 0, 1)$ the MacNeille completion $MC(P)$ of $(P; \leq)$ together with constants 0, 1 and the partial binary operation $\ominus$ defined for all $a, b \in MC(P)$ with $a \leq b$ by $b \ominus a = \wedge\{q \ominus u | u, q \in P \text{ with } u \leq a \leq b \leq q\}$ is a complete *D*-algebra.

4. *D*-ALGEBRA EMBEDDINGS AND MACNEILLE COMPLETIONS OF *D*-ALGEBRAS

Definition 4.1. Let $(P; \ominus, 0, 1)$ be a *D*-algebra and let $S \subseteq P$ have the following properties:

(i) $0, 1 \in S$.
(ii) $a \in S \Rightarrow a' \in S$.
(iii) If $a, b \in S$ with $b \ominus a$ defined in P, then $b \ominus a \in S$.

Then S with the constants 0, 1 and under the restriction of $\ominus$ to S is called a *sub-D-algebra* of P.

Definition 4.2. Let $(P; \ominus_P, 0_P, 1_P)$, $(Q; \ominus_Q, 0_Q, 1_Q)$ be *D*-algebras. A map $\varphi: P \to Q$ is called a *D-algebra embedding* iff the following conditions are satisfied:

(i) φ is an injection and $\varphi(1_P) = 1_Q$.

(ii) For $a, b \in P$, $b \ominus_P a$ is defined iff $\varphi(b) \ominus_Q \varphi(a)$ is defined, in which case $\varphi(b \ominus_P a) = \varphi(b) \ominus_q \varphi(a)$. If $\varphi(P) = Q$, then we say that D-algebras P and Q are *isomorphic*.

Evidently $\varphi(P)$ in Definition 4.2 is a sub-D-algebra of $(Q; \ominus_q, 0_Q, 1_Q)$.

Definition 4.3. We say that a complete D-algebra $(Q; \ominus_Q, 0_Q, 1_Q)$ is a *MacNeille completion of a D-algebra* $(P; \ominus_P, 0_P, 1_P)$ iff there exists a D-algebra embedding φ of P into Q such that for every $x \in Q$ there exist A, $B \subseteq P$ such that $x = \vee\varphi(A) = \wedge\varphi(B)$. In such a case we usually identify P with $\varphi(P) \subseteq Q$. If such a complete D-algebra Q does not exist, then we say that the D-algebra P does not have a MacNeille completion.

Theorem 4.4. A D-algebra $(P; \ominus, 0, 1)$ has a MacNeille completion iff P is strongly D-continuous.

Proof. (1) If D-algebra $(P; \ominus, 0, 1)$ is strongly D-continuous, then $(MC(P); \ominus, 0, 1)$, where for $a, b \in MC(P)$ with $a \leq b$ we put

$$b \ominus a = \wedge\{q \ominus u | u, q \in P \text{ such that } u \leq a \leq b \leq q\}$$

is a MacNeille completion of P, in view of Theorem 3.6.

(2) If $(Q; \ominus_Q, 0_Q, 1_Q)$ is a MacNeille completion of $(P; \ominus_P, 0_P, 1_P)$, then clearly $\varphi(P)$ is a sub-D-algebra of Q isomorphic to the D-algebra P. By Theorem 2.3 the D-algebra Q is strongly D-continuous. Since for $a, b \in P$ we have $a \leq_P b$ iff $\varphi(a) \leq_Q \varphi(b)$ and $MC(P)$ inherits all suprema and infima existing in P, we obtain that also $\varphi(P)$ and P are strongly D-continuous.

5. EFFECT ALGEBRA EMBEDDINGS AND MACNEILLE COMPLETIONS OF EFFECT ALGEBRAS

In ref. 4 the notion of *sub-effect algebra* of an effect algebra $(E; \oplus, 0, 1)$ is defined as a subset $F \subseteq E$ with properties: (i) $0, 1 \in F$, (ii) $p \in F \Rightarrow p' \in F$, and (iii) $p, q \in F$ and $p \oplus q$ is defined $\Rightarrow p \oplus q \in F$.

Notice that $F \subseteq E$ is a sub-effect algebra of $(E; \oplus, 0, 1)$ iff F is a sub-D-algebra of the D-algebra $(E; \ominus, 0,1)$ derived from $(E; \oplus, 0, 1)$. The proof follows easily from the fact that for $a, b \in F$ we have $b \ominus a = c$ iff $c' = a \oplus b'$.

Definition 5.1. Let $(E; \oplus_E, 0_E, 1_E)$, $(Q; \oplus_Q, 0_Q, 1_Q)$ be effect algebras. A map $\varphi: P \to Q$ is called an *effect algebra embedding* iff the following conditions are satisfied:

(i) φ is an injection and $\varphi(1_E) = 1_Q$.
(ii) for $a, b \in E$, $a \oplus_E b$ is defined iff $\varphi(a) \oplus_Q \varphi(b)$ is defined, in which case $\varphi(a \oplus_E b) = \varphi(a) \oplus_Q \varphi(b)$. If $\varphi(E) = Q$, we say that

effect algebras E and Q are *isomorphic*.

We can easily see that $\varphi(E)$ in Definition 5.1 is a sub-effect algebra of the effect algebra Q. Moreover, $\varphi: E \to Q$ is an effect algebra embedding iff φ is a D-algebra embedding for the derived D-algebras.

Definition 5.2. A complete effect algebra $(Q, \oplus_Q, 0_Q, 1_Q)$ is a *MacNeille completion of an effect algebra* $(E; \oplus_E, 0_E, 1_E)$ iff there exists an effect algebra embedding $\varphi: E \to Q$ such that for every $x \in Q$ there exist $A, B \subseteq E$ such that $x = \vee\varphi(A) = \wedge\varphi(B)$. We usually identify E with $\varphi(E) \subseteq Q$. If such a complete effect algebra Q does not exist, we say that the effect algebra E does not have a MacNeille completion.

As for D-algebras, the notion of a *strongly D-continuous effect algebra* $(E; \oplus, 0, 1)$ is defined by requiring E to satisfy the condition (SDC) from Definition 2.1.

Theorem 5.2. An effect algebra $(E; \oplus, 0, 1)$ has a MacNeille completion iff E is strongly D-continuous.

The proof is obvious in view of Theorem 4.4 and Section 1.

6. EXAMPLES

Example 6.1 [14]. An effect algebra $(E; \oplus, 0, 1)$ [a D-algebra $(E; \ominus, 0, 1)$] is called proper iff there exist $a, b \in E$ such that $a \oplus b$ is defined, $a \wedge b = 0$, and $a \vee b$ does not exist. Using Theorem 5.2 (Theorem 4.4), we can easily see that *proper effect algebras* (D-*algebras*) *are not strongly D-continuous and hence they do not have MacNeille completions*. This follows by putting $U = \{a, b\}$, and $Q = \{a \oplus b\}$. Then $U \leq Q$, $Q \ominus U = U$, and $\wedge(Q \ominus U) = a \wedge b = 0$, but $a \oplus b$ is not under any upper bound of A because $a \oplus b$ is not a supremum of U.

Note that an example of a proper effect algebra gives Example 2.4.

In ref. 4 the notions of a central element and the center of an effect algebra were introduced (see also ref. 15). An element $z \in E$ is called a *central element* iff for every $x \in E$ there exist $x \wedge z$ and $x \wedge z'$ and $x = (x \wedge z) \vee (x \wedge z')$. The set of all central elements of E is called a *center*, denoted by $C(E)$. It was shown in ref. 4 that $C(E)$ is always a Boolean algebra.

Example 6.2. Suppose that $(E; \oplus_E, 0_E, 1_E)$ is a lattice effect algebra [i.e., $(E; \leq_E)$ is a lattice]. Let the center $C(E)$ of E be atomic. Let $\{z_\kappa | \kappa \in H\}$ be the set of all atoms of $C(E)$. Assume that for every $\kappa \in H$ the interval $[0, z_\kappa]$ is a finite set. Then $[0, z_\kappa]$ with $\oplus$ inherited from E is an effect algebra in its own right and $\Pi_{\kappa \in H} [0, z_\kappa]$ defined "coordinatewise" is a complete effect algebra. Since $C(E)$ is a Boolean algebra, we have $\vee\{z_\kappa | \kappa \in H\} = 1$

and $z_{\kappa_1} \wedge z_{\kappa_2} = 0$ for all $\kappa_1 \neq \kappa_2$. Moreover, for every $x \in E$ there exist $x \wedge z_\kappa$, $\kappa \in H$. Let a map $\varphi: E \to \prod_{\kappa \in H} [0, z_\kappa]$ be defined by $\varphi(x) = (x \wedge z_\kappa)_{\kappa \in H}$ for all $x \in E$. Then for every $y \in \prod_{\kappa \in H}[0, z_\kappa]$ we have $y = (u_\kappa)_{\kappa \in H}$, where for $\kappa \in H$, $u_\kappa \in [0, z_\kappa] \subseteq E$ and $y = \vee\{\varphi(u_\kappa) | \kappa \in H\}$. Thus $\varphi(E)$ is supremum-dense in $\prod_{\kappa \in H}[0, z_\kappa]$. We can show that φ is an injection such that for $a, b \in E$, $a \oplus_E b$ exists iff $\varphi(a) \oplus_{MC(E)} \varphi(b)$ exists, in which case $\varphi(a \oplus_E b) = \varphi(a) \oplus_{MC(E)} \varphi(B)$. It implies that φ is an effect algebra embedding. We conclude that $\prod_{\kappa \in H} [0, z_\kappa]$ is the MacNeille completion of E. By Theorem 5.2, E is strongly D-continuous. We obtain the following assertion:

Theorem 6.3. Every lattice effect algebra with atomic center and such that there is only finite set of elements under every atom of the center is strongly D-continuous.

Example 6.4. Suppose that $(L; \vee, \wedge, \perp, 0, 1)$ is an orthomodular lattice [7]. It is well known that if we define a partial binary operation $\oplus$ on L by

for $a, b \in L$, $a \oplus b$ is defined iff $a \leq b^\perp$, in which case $a \oplus b = a \vee b$

then $(L; \oplus, 0, 1)$ is an effect algebra (derived from that orthomodular lattice L).

It is well known that the MacNeille completion of an orthomodular lattice $(L; \vee, \wedge, \perp, 0, 1)$ is always a complete ortholattice with orthocomplementation which in a natural way extends the orthocomplementation from L. But the orthomodular law is not preserved by MacNeille completion in general. Some positive results are shown in refs. 2, 6, and 10–14.

For effect algebras derived from orthomodular lattices we can prove the following theorem.

Theorem 6.5. Let $(L; \oplus, 0, 1)$ be an effect algebra derived from an orthomodular lattice $(L; \vee, \wedge, \perp, 0, 1)$ by

for $a, b \in L$, $a \oplus b$ is defined iff $a \leq b^\perp$, in which case $a \oplus b = a \vee b$

Then $(L; \oplus, 0, 1)$ has a MacNeille completion iff $MC(L)$ is an orthomodular lattice.

Proof. (1) Suppose that $MC(L)$ is a complete orthomodular lattice. Then the effect algebra derived from this complete orthomodular lattice is a complete effect algebra, which obviously is a MacNeille completion of the effect algebra derived from the orthomodular lattice L.

(2) Assume now that $(L; \oplus, 0, 1)$ has the MacNeille completion $(E; \oplus, 0, 1)$. Then evidently $E = MC(P)$ and E is a complete ortholattice, where for $a \in E$ we have

$$a^\perp = \wedge\{x^\perp | x \in L, x \leq a\} = \wedge\{1 \ominus x | x \in L, x \leq a\}$$

Suppose that $a, b \in E$ with $a \leq b$. Let $A = \{x \in L | x \leq a\}$ and $B = \{y \in$

$L | y \geq b\}$. Let $\mathscr{E} = \{\alpha \subseteq A \cup B |\ \alpha$ is finite and $\alpha \cap A \neq \emptyset \neq \alpha \cap B\}$ be directed by the set inclusion. We denote $x_\alpha = \vee\alpha \cap A$, $y_\alpha = \wedge\alpha \cap B$ for every $\alpha \in \mathscr{E}$ Then $x_\alpha \uparrow a$, $y_\alpha \downarrow b$. Moreover, for every $\alpha \in \mathscr{E}$ we have $x_\alpha \leq y_\alpha$ and hence $y_\alpha = x_\alpha \vee (x_\alpha^\perp \wedge y_\alpha)$. Since $x_\alpha^\perp \downarrow a^\perp$ and $y_\alpha \downarrow b$, we obtain $x_\alpha^\perp \wedge y_\alpha \downarrow a^\perp \wedge b$, which by ref. 13 implies that $a^\perp \wedge b = b \ominus a$. Since $a \wedge (a^\perp \wedge b) = 0$, we obtain by ref. 4, Theorem 3.5, that $b = a \vee (a^\perp \wedge b)$.

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